The Algebraic Frequency Equation of a Taut String Damped at a Fraction Location: Solution Structures and Properties

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Abstract. The frequency equation for a cable damped at a fraction location, generally being a transcendental equation, was reformulated into an algebraic form. Based on the fundamental theorem of algebra and the characteristics of logarithmic function in the complex domain, the solution structures were revealed and several examples with given damper positions were presented to study the variation of the solutions with the damping coefficient. The results show that: 1) All solutions of the frequency equation of the system could be classified into a finite number of solution branches. 2) The solutions of different orders in the same solution branch share an identical real part and any two adjacent solutions share an identical difference. 3) According to different variations in decay rate and frequency with damping coefficient, all the solution branches could be classified into four categories, each of which showed different trend of variation.

Keywords: cable-damper system · taut string · solution structures · solution branch · decay rate

1 Introduction

Vibration mitigation of stay cables has attracted much attention in bridge engineering research (Johnson et al. 2007; Chen et al. 2020; Chen et al. 2022). A common way to reduce the vibration of cables in engineering practice is to use the external viscous dampers (Yoneda and Maeda 1989; Takano et al. 1997; Sun et al. 2001; Dyke et al. 2003). The early researches on control of cable vibration using viscous dampers mainly focused on the lower modes. However, with the development of long-span cable-stayed bridges and the length of stay cables becoming longer and longer, the higher order modes may dominate the vibration of cables (Gao et al. 2018; Chen et al. 2019; Yang et al. 2021; Kim et al. 2022; Yang et al. 2022), which necessitates researches of both lower and higher modes vibration of cable-damper system.

In existing studies, the cable-damper system is usually simplified as a taut string with lumped damping (Tabatabai et al. 2000; Hoang et al. 2008; Chen et al. 2004;

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Pacheco et al. 1993; Krenk et al. 2000; Main et al. 2002 and Zheng et al. 2020). And the complex eigenvalue problem of the system with different damper positions and damping coefficients is studied to evaluate the vibrational characteristics of the system. Pacheco (1993) obtained the numerical solution of the complex eigenvalue problem and developed the approximate expression of the modal-damping ratios for the first few modes for the damper positions near the anchorage. Krenk (2000) derived a general transcendental frequency equation and obtained approximate analytical solutions, resulting in another approximate expression of the lower modal damping ratios while the damper is near the end of cables. For the higher modes, Yang (2021) improved the solution method of Krenk and obtained an approximate analytical solution of the higher modal damping ratios with the same damper positions. It is worth noting that Main (2002) derived the same complex frequency equation with Krenk and analyzed the variations of frequencies and modal-damping ratios of the system vibration of all orders with the damper position and damping coefficient.

In fact, the frequency equation of the cable-damper system is a transcendental equation without general analytical solution methods. In the above studies, numerical or approximate analytical methods had been used to obtain the eigen-solutions of the system and remarkable accomplishments had been achieved in dealing with engineering problems with these results. However, the relationship between the low and high order eigenvalues of the system is not easily revealed in the above solutions. In order to get more insight into the dynamic characteristics of the system, it is of great significance to find a fully analytical solution method of the frequency equation.

Zheng (2021) obtained the algebraic form of the frequency equation and got the first analytical solution of the system for the special case that the damper is located at the middle span of the cable. This paper borrowed the idea from Zheng (2021) to simplify the frequency equation into an algebraic form for the case the damper is located at any fraction position of the cable. Based on the algebraic frequency equation and its analytical solution, the structural characteristics and properties of the eigen-solutions are discussed to provide a new perspective for understanding the dynamic characteristics of the cable-damper system.

2 Transcendental Form of Frequency Equation

A taut string with a concentrated viscous damper is shown in Fig. 1, where $\bar{l}$ denotes length, $\bar{m}$ denotes mass per unit length, $\bar{T}$ denotes tension, $\bar{c}$ denotes the damping coefficient of the damper and $\bar{x}_{\mathrm{d}}$ denotes the distance from the damper to the left end of the string ($0 < \bar{x}_{\mathrm{d}} < \bar{l}$).

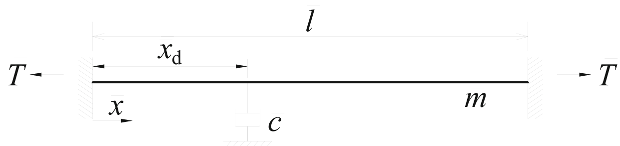

Fig. 1. A taut string with a lumped damping

The equation of motion of the system could be described by the following partial differential equation

$$\bar{T}\frac{\partial^2\bar{w}(\bar{x},\bar{t})}{\partial\bar{x}^2} - \bar{m}\frac{\partial^2\bar{w}(\bar{x},\bar{t})}{\partial\bar{t}^2} = \bar{c}\frac{\partial\bar{w}(\bar{x},\bar{t})}{\partial\bar{t}}\bar{\delta}(\bar{x}-\bar{x}_{\mathrm{d}}) \tag{1}$$

where $\bar{x}$ is the coordinatesx, $\bar{t}$ is time, $\bar{w}(\bar{x},\bar{t})$ is the transverse deflection and $\bar{\delta}(\bar{x}-\bar{x}_{\mathrm{d}})$ is Dirac Delta Function. In Eq. (1), an overbar represents dimensional physical quantities. Referring to Zheng (2021), the following transformations are introduced

$$\bar{t} = t\sqrt{\bar{m}\bar{l}^2/\bar{T}} = t/\bar{\omega} \tag{2}$$

$$\bar{\omega} = \sqrt{\bar{T}/\bar{m}}/l \tag{3}$$

$$x = \bar{x}/\bar{l} \tag{4}$$

$$\delta(x-x_{\mathrm{d}}) = \bar{l}\bar{\delta}(\bar{x}-\bar{x}_{\mathrm{d}}) \tag{5}$$

$$c = \bar{c}/\sqrt{\bar{m}\bar{T}} \tag{6}$$

$$\bar{w} = \bar{l}w \tag{7}$$

Substituting Eq. (2)–(7) into Eq. (1), the equation of motion is transformed into a fully nondimensionalized form.

$$-\frac{\partial^2 w(x,t)}{\partial x^2} + \frac{\partial^2 w(x,t)}{\partial t^2} = -\frac{\partial w(x,t)}{\partial t}c\delta(x-x_{\mathrm{d}}) \tag{8}$$

In Eq. (8), c denotes the dimensionless damping coefficient, x_{d} denotes the dimensionless damper position, $0 < x_{\mathrm{d}} < 1$.

Based on Eq. (8), the eigenvalue equation of the nondimensionalized system, also known as the frequency equation, is derived

$$\frac{\sinh(p(1-x_{\mathrm{d}}))\sinh(px_{\mathrm{d}})}{\sinh(p)} = -\frac{1}{c} \tag{9}$$

Equation (9) is a complex transcendental equation and each solution to Eq. (9), corresponding to a complex eigenvalue of the system could be written explicitly in terms of real and imaginary parts as $p = \sigma + j\omega$, where $-\sigma$ is the decay rate of the system vibration and ω is the natural frequency.

3 Algebraization of Frequency Equation

Equation (9) could usually be solved by numerical method or approximate method, since it is a transcendental equation. However, for the special cases that the parameter xd takes any rational number, xd = m/n (m, n∈N+), it turned out that Eq. (9) could be transformed into an algebraic equation. As a result, the structure of the eigen-solutions could be analyzed by using the basic theorem of algebra and other related theories, and the properties of the eigen-solutions could be explored by means of analytical examples.

It should be pointed out that due to symmetry of the system about its midpoint, it is only necessary to discuss the case where the damper is located on the left half of the string. Therefore, we can assume $x_\mathrm{d} \leq 1/2$. At the same time, let x_d be a simplest fraction $x_\mathrm{d} = m/n$.

Substitute the expression $x_\mathrm{d} = m/n$ into Eq. (9) and expand the hyperbolic function in terms of exponential functions

$$\frac{\left(\mathrm{e}^{\frac{p}{n}(n-m)} - \mathrm{e}^{-\frac{p}{n}(n-m)}\right)\left(\mathrm{e}^{\frac{p}{n}m} - \mathrm{e}^{-\frac{p}{n}m}\right)}{2\left(\mathrm{e}^{p} - \mathrm{e}^{-p}\right)} = -\frac{1}{c} \tag{10}$$

Let $y = e^{2p/n}$, Eq. (10) can be simplified into an n-th degree polynomial equation about y

$$\left(1 + \frac{2}{c}\right)y^n - y^{n-m} - y^m + \left(1 - \frac{2}{c}\right) = 0 \tag{11}$$

Equation (11) could easily be solved by the fundamental theorem of algebra and then by the relationship $\mathrm{y} = \mathrm{e}^{2p/n}$, the eigenvalue p of the system could be written as

$$p = \frac{n}{2}\ln(y), y \neq 1 \tag{12}$$

Equation (11) could be deemed as the algebraic frequency equation of the system of the taut string with a lumped damping at a fraction location. And its solution could be used to give the complex eigenvalue of the system by Eq. (12). It is worth pointing out that the trivial complex eigenvalue $p = 0$ corresponding to $y = 1$ is omitted from the subsequent discussion.

4 Structure of Eigen-Solutions

4.1 System Eigen Solution Branches

Note that the eigenvalue given by Eq. (12) is written as a logarithmic function, which has multi-valued characteristics in the complex field. As a result, one root y of Eq. (11) corresponds to a set of eigen value ps. Based on this relationship, all the eigenvalues of the system could be classified into several eigen-solution branches, each of which corresponds to one root of Eq. (11). The number and the structural characteristics of solution branches will be explored in the following.

According to the basic theorem of algebra, the frequency equation Eq. (11) has n roots and only $n - 1$ of them need to be considered after discarding the trivial root of $y = 1$. Substituting the n-1 roots of Eq. (11) into Eq. (12), we get $n - 1$ groups of complex eigenvalues and each group corresponds to one eigen-solution branch. Based on the above results, the solutions to Eq. (12) can be written as

$$p^{(i)} = \frac{n}{2}\ln\left(y^{(i)}\right), i = 1, 2, \cdots n - 1 \tag{13}$$

where $y^{(i)}$ is the i-th root of Eq. (11) and $p^{(i)}$ is the i-th eigen-solutions branch.

It is noted the number of eigen-solution branches of the system is equal to $n - 1$ and merely depends on the damper position.

4.2 Decay Rates and Frequencies of System Vibrations

According to the relevant formula of logarithmic function, the real and imaginary parts of the eigen-solutions Eq. (13), corresponding to the decay rates and frequencies of system vibrations, respectively, could be written as

$$\sigma^{(i)} = \mathrm{Re}\left(p^{(i)}\right) = \frac{n}{2}\ln|y^{(i)}| \tag{14}$$

$$\omega^{(i)} = \mathrm{Im}\left(p^{(i)}\right) = \frac{n}{2}\arg(y^{(i)}) + ns\pi,\ s = 0, \pm 1, \pm 2, \cdots \tag{15}$$

where arg() denotes argument function.

Decay Rates of System Vibrations. It can be seen from Eq. (14) that for a string system damped at a fraction location, the real parts of the eigenvalues of different order under the same solution branch are identical. Therefore, the system motions characterized by these eigenvalues share identical decay rate and the amplitude of each mode of vibration will decay according to the same exponential function.

Frequencies of System Vibrations. It can be seen from Eq. (15) that the imaginary parts of eigenvalues of different order under the same solution branch form an arithmetic sequence with a common difference. It is noted that this property resembles that of the undamped string.

Based on the structural characteristics of eigenvalues under the same solution branch presented above, it could be concluded that as long as one eigen-solution of any solution branch of the system is obtained, all the other eigen-solutions in the same solution branch are known.

5 Properties of Eigen-Solutions

It could be seen from Eq. (11) that when the damper position parameter $x_\mathrm{d} = m/n$ is given, the eigen-solutions will generally vary with the damping coefficients. In what follows, analytical eigen-solutions of the system for several given damper positions will be presented to study the variation of eigenvalues with the damping coefficient.

It is noted that when $n < 6$, Eq. (11) is a polynomial equation of degree 4 or below, which has a radical solution and may help to get further insight into the dynamics of the system. Let $x_\mathrm{d} = 1/2$, 1/3, 1/4, 1/5 and 2/5, respectively. These different values of x_d represent different damper-string systems. Substitute x_d into Eq. (11) and solve it. Substituting the roots to Eq. (11) into Eq. (12), the closed-form solution of the eigenvalues could be obtained. Based on these eigen-solutions, the variation of the decay rates and frequencies of the system vibrations with the damping coefficients could be studied and the results were shown in Figs. 2, 3, 4, 5 and 6, respectively.

In addition, referring to Main (2002), the system with damper position $x_\mathrm{d} = 3/7$ may take on different dynamic characteristics as the above systems. So, the system with $x_\mathrm{d} = 3/7$ is also considered. Note that for $x_\mathrm{d} = 3/7$, no radical solution to Eq. (11) exists and only numerical solutions could be obtained. The results are shown in Fig. 7. It is worth pointing out that there is only one system with the damper located at a non-unit fraction location $x_\mathrm{d} = 2/5$, which could be studied analytically by radical solutions to Eq. (11).

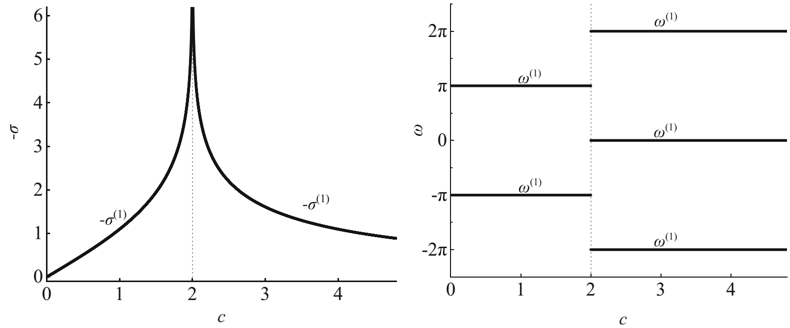

Fig. 2. The taut string with a lumped damping at middle-span. (left) The variation of decay rates of system vibration with damping coefficients. (right) The variation of frequencies of system vibration with damping coefficient.

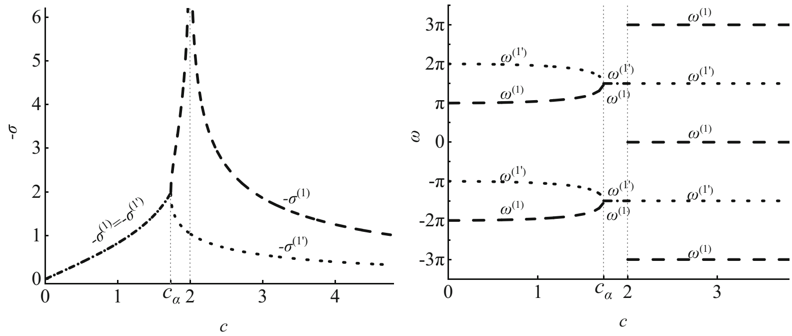

Fig. 3. The taut string with a lumped damping at one-third-span. (left) The variation of decay rates of system vibration with damping coefficient. (right) The variation of frequencies of system vibration with damping coefficient.

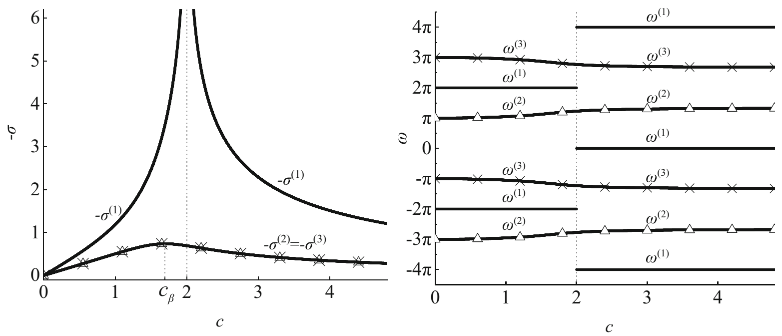

Fig. 4. The taut string with a lumped damping at one-fourth-span. (left) The variation of decay rates of system vibration with damping coefficient. (right) The variation of frequencies of system vibration with damping coefficient.

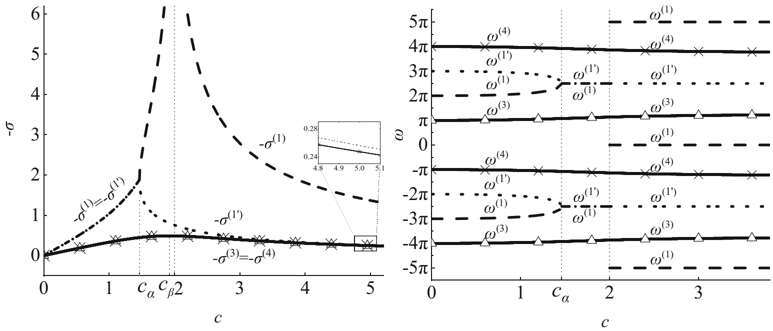

Fig. 5. The taut string with a lumped damping at one-fifth-span. (left) The variation of decay rates of system vibration with damping coefficient. (right) The variation of frequencies of system vibration with damping coefficient.

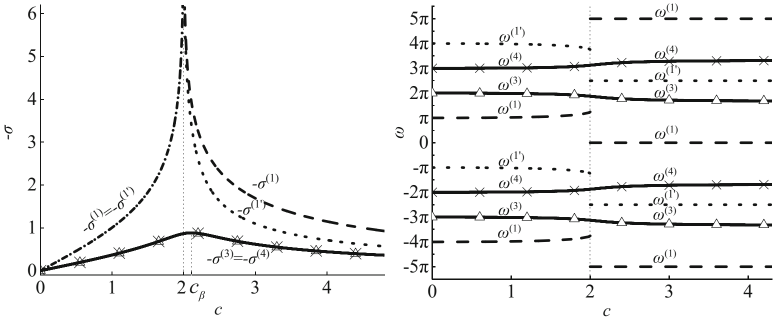

Fig. 6. The taut string with a lumped damping at two-fifth-span. (left) The variation of decay rates of system vibration with damping coefficient. (right) The variation of frequencies of system vibration with damping coefficient.

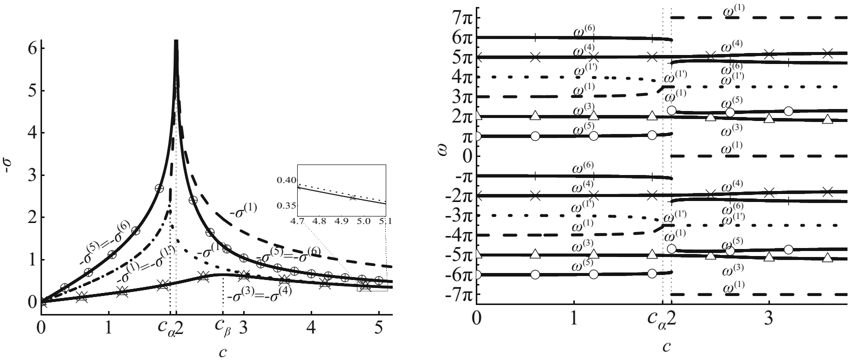

Fig. 7. The taut string with a lumped damping at three-seventh-span. (left) The variation of decay rates of system vibration with damping coefficient. (right) The variation of frequencies of system vibration with damping coefficient.

5.1 Variational Characteristics of Solution Branches with Damping Coefficients

It could be observed from Figs. 2, 3, 4, 5, 6 and 7 that all systems have a special solution branch, which is marked as solution branch $p^{(1)}$ ($-\sigma^{(1)}$ *and* $\omega^{(1)}$). The corresponding decay rate of this special solution branch monotonically increases from 0 to infinity as the damping coefficient increases from 0 to 2 and monotonically decreases from infinity to 0 as the damping coefficient increases from 2 to infinity. When the damping coefficient is equal to 2, the decay rate tends to infinity and the motions represented by the solution branch $p^{(1)}$ decay to zero instantaneously. The damping coefficient of 2 is called the critical damping coefficient by Main 2002. It could also be seen from Figs. 2, 3, 4, 5, 6 and 7 that when the damping coefficient gradually increases from 0 to infinity, the decay rate of all vibrations of any system monotonically increases from 0 to its maximum value and then monotonically decreases to 0. In the following, in discussing the relationship between decay rate and damping coefficient, we will mainly focus on the maximum decay rate of each solution branch and its corresponding damping coefficient.

It could also be seen from Figs. 2, 3, 4, 5, 6 and 7 that the frequencies of the system vibration corresponding to the special solution branch $p^{(1)}$ are constants within a certain range of damping coefficient. When $c \geq 2$, the constants are $sn\pi$. Note that when s is 0, non-oscillatory decay of the system will occur. When the damping coefficient $c <$ 2, the vibration frequencies corresponding to the solution branch $p^{(1)}$ vary differently for different damper positions: 1) When n is an even number (such as the system of x_{d} = 1/2), the corresponding frequencies of the solution branch take constant values (s + 1/2) $n\pi$; 2) When n is an odd number and m is an even number (such as the case of $x_{\mathrm{d}} = 2/5$), the corresponding frequencies of the solution branch vary with the damping coefficient; 3) When n and m are both odd numbers (such as $x_{\mathrm{d}} = 1/3$ or 3/7), the vibration frequencies corresponding to branches $p^{(1)}$ vary with damping coefficient when it is less than a certain value c_α ($c_\alpha < 2$). When $c_\alpha < c < 2$, the frequencies again take constants (s + 1/2) $n\pi$.

When n is an odd number, there is another special solution branch for the system, which is denoted as $p^{(1')}$ instead of $p^{(2)}$, see Figs. 3 and 5, 6, 7. This solution branch shows different variational characteristics for different damper positions: 1) When m is an even number (corresponding to the two-fifth-span system, see Fig. 6), the decay rate of the system vibrations tends to infinity when the damping coefficient approaches 2. The frequencies of the system vibrations vary with the damping coefficient when the damping coefficient $c < 2$ and are constants (s + 1/2) $n\pi$ when the damping coefficient $c \geq 2$, which is different from that of $p^{(1)}$ solution branch; 2) When m is an odd number (such as the one-third-span system, see Fig. 3), the decay rate of the system vibration attains a local maximum value when the damping coefficient is $c\alpha$. The frequencies of the system vibrations vary with the damping coefficient when $c < c_\alpha$ and are constants (s + 1/2)$n\pi$ when $c \geq c_\alpha$.

It is worth noting that only when n is odd, the system has both special solution branches $p^{(1)}$ and $p^{(1')}$. For the case that n is odd and m is even, the solution branches $p^{(1)}$ and $p^{(1')}$ are conjugate to each other when $c < 2$, which indicates the corresponding vibrations of the system shares identical decay rates and frequencies. When $c \geq 2$, the two solution branches are no longer conjugate and the decay rates and frequencies of the corresponding vibrations are not the same. For the system that n and m are both odd numbers, the solution branch $p^{(1)}$ and $p^{(1')}$ are conjugate with each other only when $c < c_\alpha$. When $c_\alpha \leq c < 2$, the decay rates of the vibrations corresponding to the solution branches $p^{(1)}$ and $p^{(1')}$ show different variations with the increase of c while their frequencies are equal constants, see Fig. 3 for an example. When $n \geq 2$, the frequencies of the vibrations corresponding to the two solution branches are still constant, but their values are different.

For the systems with $n \geq 4$, a new kind of solution branch could be found (such as $p^{(2)}$ and $p^{(3)}$of the one-fourth-span system, see Fig. 4). The corresponding decay rates change continuously with the damping coefficient and attain a local maximum value when the damping coefficient takes a certain value c_β ($c_\beta \neq c_\alpha$). The corresponding frequencies also vary continuously with the damping coefficient. It is noted that for any damping coefficient, all solutions in such a solution branch appear in the form of conjugate pairs.

For systems with $m \geq 3$ and $n \geq 7$ (such as $p^{(5)}$ and $p^{(6)}$ for three-seventh-span systems, see Fig. 7), another new type of solution branch could be found. Similar to the special solution branch $p^{(1)}$,the corresponding decay rate of this kind of solution branches also tends to infinity when the damping coefficient approaches 2. However, the corresponding frequencies of this kind of solution branch vary with the damping coefficient for any value of c and show a jump at the point where the damping coefficients is 2. Note that all solutions in this kind of solution branch also appear in the form of conjugate pairs.

5.2 Classification Discussion of Solution Branches

Based on the above discussion on the solution branches, all the solution branches could be classified into four categories based on different variations of the corresponding frequencies and decay rates with the damping coefficient. The first kind of solution branches: the corresponding frequencies vary continuously with the damping coefficient and the decay rates have a local maximum value (such as $p^{(2)}$ and $p^{(3)}$ of the one-fourth-span system, see Fig. 4); The second kind of solution branches: the corresponding frequencies also vary with the damping coefficient but show a jump discontinuity and the maximum value of the decay rate is infinite (for example, $p^{(5)}$ and $p^{(6)}$ of the three-seventh point system, see Fig. 7); The third kind of solution branches: the corresponding frequencies are constant and the corresponding decay rates approaches infinity at the non-differentiable point (such as $p^{(1)}$ of the middle-span system, see Fig. 2). The fourth kind of solution branches: the corresponding frequencies change continuously with the damping coefficient when it is less than a certain value and is constant when it is greater than the value. There are only two such solution branches and they only appear when n is odd. When m is an even number (e.g. $p^{(1)}$ and $p^{(1')}$ for the two-fifth-span system, see Fig. 6), the maximum values of the corresponding decay rates of both the solution branches are infinite. When m is also odd (e.g., $p^{(1)}$ and $p^{(1')}$ for the one-third-span system, see Fig. 3), the decay rate of one branch has a finite maximum value and that of the other branch has a maximum value of infinity.

6 Conclusion

In this paper, based on the normalized form of the lumped damping string model, the transcendental frequency equation of the system was reformulated into an algebraic equation when the damping was located at a fraction position of the cable. And all the eigen-solutions of the system could be classified into a finite number of solution branches, each of which corresponds to one root of the equation. Several analytical examples with given damper positions were presented and the variation of the system eigen-solutions with the damping coefficient were discussed. The main conclusions were as follows:

(1) The eigen-solutions of the system could be classified into $n - 1$ solution branches, where the number n is the denominator of the fraction m/n to represent the damper position.
(2) The eigen-solutions in the same branch share an identical real part, which indicates the corresponding vibrations of the system decay identically in time. And any two adjacent solutions in the same solution branch share an identical difference in their imaginary parts, indicating the corresponding vibrations have the same frequency difference.
(3) According to different variations in the corresponding decay rates and frequencies with the damping coefficient, all the solution branches could be classified into four categories. It was worth noting that the four kinds of solution branches share a common feature that as the damping coefficient c increases from zero to infinity, the corresponding decay rates increases monotonically from zero to their maximum and then decreases monotonically to zero. The difference between different solution branches are:

The first kind: The corresponding frequencies vary continuously with the damping coefficient and the decay rates could attain a local maximum as c takes c_β. The solution branches of this kind correspond to the conjugate complex roots of the algebraic frequency equation and always appear in the form of conjugate pairs and only appear as $n \geq 3$.

The second kind: The corresponding frequencies also vary with the damping coefficient but with a jump at the point $c = 2$ and the corresponding decay rates tends to infinity at this point. These solution branches also appear in the form of conjugate pairs corresponding to the conjugate complex roots of the algebraic equation, but only appear as $m \geq 3$ and $n \geq 7$.

The third kind: the corresponding frequencies are constant in two different damping intervals, one lying in $c \leq 2$ and the other lying in $c > 2$, but the constant values are different. The corresponding decay rates approach infinity as c takes 2. Only one such solution branch appear as n is even, corresponding to the real root of the algebraic equation.

The fourth kind: the corresponding frequencies vary with the damping coefficients in the range $0 < c \leq \beta$, corresponding to the conjugate complex roots of the algebraic frequency equation and are constants in the range $c > \beta$, corresponding to the real root of the algebraic frequency equation. There are totally two such kind of solution branches and they only appear when n is an odd number. When m is even, the corresponding decay rates of the two solution branches approach infinity at $c = \beta = 2$. When m is odd, the decay rate of one solution branch attains a finite maximum at $c = \beta = c_\alpha$ ($c_\alpha < 2$) and the decay rate of the other branch approaches infinity at $c = 2$.

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